

APPROXIMATE DIAGONALIZATION OF HOMOMORPHISMS FROM $C(X)$ TO C^* -ALGEBRAS OF TRACIAL RANK ONE

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ABSTRACT. In this paper, we prove that unital homomorphisms from continuous functions on a compact metric space to matrices over a C^* -algebra with tracial rank at most one are approximately diagonalizable.

1. INTRODUCTION

One of the fundamental facts in linear algebra is that normal matrices over \mathbb{C} are unitarily equivalent to diagonal matrices. Furthermore, any set of commuting normal matrices is simultaneously diagonalizable. Given the importance of matrices over C^* -algebras, it is a natural question to ask whether a normal matrix with entries in a C^* -algebra is diagonalizable.

Richard Kadison demonstrated in [5] that when the C^* -algebra is a von Neumann algebra any commuting set of normal matrices is simultaneously diagonalizable. Kadison also posed the question: for what topological spaces X is every normal matrix over $C(X)$ diagonalizable? Karsten Grove and Gert Pedersen answer this question in [3] that X must, among other topological restrictions, be sub-Stonian. This suggests that diagonalization for normal matrices with entries in general C^* -algebras is rather restricted. Indeed, Kadison's proof is based on the abundance of projections in maximal abelian subalgebras of von Neumann algebras, which can be extended to show diagonalization in C^* -algebras with similarly behaving maximal abelian subalgebras such as AW^* -algebras, as seen in [4], but does not generalize to larger classes of C^* -algebras.

If we instead consider approximate diagonalization, then the situation improves. For example, Yifeng Xue demonstrates in [13] that every self-adjoint matrix over $C(X)$ is approximately diagonalizable if $\dim(X) \leq 2$ and $\check{H}^2(X, \mathbb{Z}) = 0$. Further, Huaxin Lin demonstrates in [7] that if X is locally an absolute retract and Y has $\dim(Y) \leq 2$, then every unital homomorphism from $C(X)$ to $M_n(C(Y))$ is approximately diagonalizable.

As Lin points out in [7], if a is a normal element in $M_n(A)$ for some unital C^* -algebra A , then continuous functional calculus induces a unital homomorphism $\phi: C(\text{sp}(a)) \rightarrow M_n(A)$ and approximate diagonalization of a is equivalent to the approximate diagonalization of the induced homomorphism ϕ . Similarly if S is a set of commuting normal matrices in $M_n(A)$, then since $C^*(S, 1) \cong C(X)$ for some compact, Hausdorff space X , there is a unital homomorphism $\phi: C(X) \rightarrow M_n(A)$ obtained from the inclusion $C^*(S, 1) \subseteq M_n(A)$ and approximate diagonalization of the homomorphism ϕ is equivalent to the simultaneous diagonalization of finite subsets of S .

In this paper, we prove the approximate diagonalization of unital homomorphisms from $C(X)$ to matrices over C^* -algebras with tracial rank one using Lin's classification of unital, injective homomorphisms from $C(X)$ to C^* -algebras with tracial rank one. In section 2, we review the invariants used in Lin's classification theorems. In section 3, we prove that positive group homomorphisms are decomposable and that they can be decomposed to preserve commutative diagrams. In section 4, we prove the aforementioned result.

2. PRELIMINARIES

We use the notation found in [8]. In particular, if A is a unital C^* -algebra we denote $T(A)$ as the space of tracial states of A and $\text{Aff}(T(A))$ as the partially ordered vector space of real, continuous, affine functions on $T(A)$. There is a natural pairing between $K_0(A)$ and $T(A)$, which we describe with a normalized, positive group homomorphism $\rho_A: K_0(A) \rightarrow \text{Aff}(T(A))$ defined by $\rho_A([p]) = \tau \otimes \text{Tr}(p)$ for $p \in M_\infty(A)$. For convenience, we write τ for $\tau \otimes \text{Tr}$. Given another unital C^* -algebra C and a unital homomorphism from C to A , by naturality, we induce a commutative square from this pairing. On the other hand, if we have a normalized, positive group homomorphism $\alpha: K_0(C) \rightarrow K_0(A)$ and a unital, positive linear map $\gamma: \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$, we say α and γ are *compatible* if they form a commutative square: $\rho_A \circ \alpha = \gamma \circ \rho_C$.

$KL(C, A)$ is the quotient of $KK(C, A)$ by the group of pure extensions of $K_*(C)$ by $K_{*+1}(A)$. See Section 2.4.8 of [11] for more information. Since the KL -group is a quotient of the KK -group we have a KL version of the UCT (see equation 2.4.9 of [11]) when C satisfies the UCT:

$$0 \rightarrow \text{ext}(K_*(C), K_{*+1}(A)) \xrightarrow{\varepsilon} KL(C, A) \xrightarrow{\Gamma} \text{Hom}(K_*(C), K_*(A)) \rightarrow 0.$$

We denote by $KL_e(C, A)^{++}$ the set of $\kappa \in KL(C, A)$ so that $\Gamma(\kappa)(K_0(C)^+ \setminus \{0\}) \subseteq K_0(A) \setminus \{0\}$ and $\Gamma(\kappa)(1_C) = 1_A$. We say $\kappa \in KL_e(C, A)^{++}$ and a unital positive linear map $\gamma: \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$ are compatible if the restriction of $\Gamma(\kappa)$ to $K_0(C)$ and γ are compatible.

Notice that for any compact metric space X , the range of $\rho_{C(X)}$ is isomorphic to $C(X, \mathbb{Z})$. Furthermore, the short exact sequence:

$$0 \rightarrow \ker \rho_{C(X)} \rightarrow K_0(C(X)) \rightarrow C(X, \mathbb{Z}) \rightarrow 0$$

is split. In fact, $C(X, \mathbb{Z})$ is a free abelian group. This is apparent in the case when X has finitely many connected components, where $C(X, \mathbb{Z})$ is generated by the characteristic functions of the connected components of X . Furthermore, by considering extremal traces, we see that $\text{Aff}(T(C(X))) \cong C(X)_{\text{sa}}$.

We denote the group of unitaries of C by $U(C)$, the normal subgroup of the connected component containing 1_C by $U_0(C)$, the normal subgroup generated by the commutators of $U(C)$ by $CU(C)$, and $CU_0(C) = CU(C) \cap U_0(C)$. We also define $U^\infty(C) = \bigcup_{n=1}^\infty U(M_n(C))$, and similarly define $U_0^\infty(C)$, $CU^\infty(C)$, and $CU_0^\infty(C)$. Also for every unitary $u \in U(C)$, we denote the equivalence class in $U(C)/\overline{CU(C)}$ containing u as \bar{u} .

For each $\tau \in T(C)$, $u \in U_0^\infty(C)$, and each piecewise smooth path $\zeta \in C([0, 1], U_0^\infty(C))$ with $\zeta(0) = 1$ and $\zeta(1) = u$, we define

$$\Delta(\zeta)(\tau) = \int_0^1 \text{Tr} \otimes \tau \left(\frac{d\zeta(t)}{dt} \zeta^{-1}(t) \right) dt.$$

As shown in [12], this induces a continuous homomorphism

$$\bar{\Delta}: U_0^\infty(C)/\overline{CU^\infty(C)} \rightarrow \text{Aff}(T(C))/\overline{\rho_C(K_0(C))}$$

As seen in [12], this provides a natural short exact sequence:

$$0 \rightarrow \text{Aff}(T(C))/\overline{\rho_C(K_0(C))} \rightarrow U^\infty(C)/\overline{CU^\infty(C)} \rightarrow K_1(C) \rightarrow 0$$

This short-exact sequence is split, though unnaturally. We denote by π_C the quotient map $U^\infty(C)/\overline{CU^\infty(C)} \rightarrow K_1(C)$.

Given a unital homomorphism $\phi: C \rightarrow A$, we denote by the induced continuous homomorphism

$$\phi^\dagger: U^\infty(C)/\overline{CU^\infty(C)} \rightarrow U^\infty(A)/\overline{CU^\infty(A)}.$$

Consider a compatible pair $\kappa \in KL_e^{++}(C, A)$ and $\gamma: \text{Aff}(T(C)) \rightarrow \text{Aff}(T(A))$.

Let $\eta: U^\infty(C)/\overline{CU^\infty(C)} \rightarrow U^\infty(A)/\overline{CU^\infty(A)}$ be a continuous homomorphism. If the restriction of η to $\text{Aff}(T(C))/\overline{\rho_C(K_0(C))}$ is equal to the homomorphism induced from γ and the restriction of η and κ to $K_1(C)$ are equal, then we say that κ , γ , and η are *compatible*.

3. DECOMPOSITION IN PARTIALLY ORDERED ABELIAN GROUPS

As a group, $K_0(C(X))$ can be written as the inductive limit of finitely generated abelian groups and so it is a relatively straightforward matter to define homomorphisms from $K_0(C(X))$. The ordering of $K_0(C(X))$ is not easily determined since topological difficulties may lead to perforation. Fortunately, if the target of the homomorphism has an ordering determined by its states, then these pathologies can be managed and we can define positive group homomorphisms.

We adopt some language about partially ordered abelian groups from [2]. Following [2], we will refer to partially ordered abelian groups with the Riesz interpolation property as interpolation groups.

Definition 3.1. Let S be a partially ordered set with a least element 0. An element $x \in S$ is called an *atom* of S if there is no element $y \in S$ for which $0 < y < x$.

When X is topological space with finitely many connected components, the characteristic functions of those components are the atoms of $K_0(C(X))^+$ and generate the subgroup $C(X, \mathbb{Z})$. Based on the comment in Section 2, we see that when X has finitely many connected components, $K_0(C(X))$ can be decomposed as a group into the direct sum of $C(X, \mathbb{Z})$, the subgroup generated by the atoms, and $\ker \rho_{C(X)}$, the intersection of the kernels of the states of $(K_0(C(X)), 1_{C(X)})$.

We define the group homomorphism $\rho_G: (G, u) \rightarrow (\text{Aff}(S(G, u)), 1)$ by $\rho_G(g)(\sigma) = \sigma(g)$. We note that the intersection of the kernel of the states of (G, u) is equal to $\ker \rho_G$. Also when C is exact, $\rho_C = \rho_{K_0(C)}$.

In contrast, when a partially ordered abelian group G is simple, G^+ contains no atoms except when G is cyclic (Lemma 14.2 of [2]). It will be useful to treat \mathbb{Z} separately. For example, when G is a non-cyclic, simple interpolation group, G also satisfies a strict version of interpolation (Proposition 14.6 of [2]).

Definition 3.2. A partially ordered abelian group G has *strict interpolation* if for all $x_1, x_2, y_1, y_2 \in G$ such that $x_i < y_j$ for all i, j , there exists $z \in G$ such that $x_i < z < y_j$ for all i, j .

Strict versions of the Riesz decomposition properties follow with analogous proofs. See, for example, Propositions 2.1 and 2.2 of [2].

Proposition 3.3. *Let G be a partially ordered abelian group. The following are equivalent:*

- (a) *G has strict interpolation.*
- (b) *If $x, y_1, y_2 \in G$ satisfying $0 < x < y_1 + y_2$, then there exist $x_1, x_2 \in G^+ \setminus \{0\}$ such that $x_1 + x_2 = x$ and $x_i < y_i$ for $i = 1, 2$.*
- (c) *If $x_1, x_2, y_1, y_2 \in G^+ \setminus \{0\}$ satisfying $x_1 + x_2 = y_1 + y_2$, then there exist $z_{i,j} \in G^+ \setminus \{0\}$ such that $x_i = z_{i,1} + z_{i,2}$ and $y_j = z_{1,j} + z_{2,j}$ for $i = 1, 2, j = 1, 2$.*

Proposition 3.4. *Let G be a partially ordered abelian group with strict interpolation. Then the following hold:*

- (a) *If x_1, x_2, \dots, x_n and y_1, \dots, y_k are in G such that $x_i < y_j$ for all i, j , then there exists $z \in G$ such that $x_i < z < y_j$ for all i, j .*
- (b) *If $x, y_1, y_2, \dots, y_n \in G^+ \setminus \{0\}$ satisfying $x < y_1 + y_2 + \dots + y_n$, then there exist $x_1, \dots, x_n \in G^+ \setminus \{0\}$ such that $x = x_1 + \dots + x_n$ and $x_i < y_i$ for all i .*
- (c) *If $x_1, \dots, x_n, y_1, \dots, y_k \in G^+ \setminus \{0\}$, then there exists $z_{i,j}$ for $i = 1, 2, \dots, n, j = 1, 2, \dots, k$ such that $x_i = z_{i,1} + \dots + z_{i,k}$ and $y_j = z_{1,j} + \dots + z_{n,j}$.*

When a partially ordered abelian group (G, u) is simple and weakly unperforated, the order on G is determined by its states. Namely for all $x \in G$, $x \geq 0$ if and only if $\sigma(x) \geq 0$ for all $\sigma \in S(G, u)$. In fact, since $\sigma(x) = 0$ only when $x = 0$ by simplicity, we see that $x > 0$ if and only if $\sigma(x) > 0$ for all $\sigma \in S(G)$. But when considering C^* -algebras, we are more concerned when the order on K_0 is determined by the tracial states, rather than the quasi-traces. With this in mind, we make the following definition.

Definition 3.5. Let (G, u) be a partially ordered abelian group and let $T \subseteq S(G, u)$ be a non-empty subset. The *order of G is determined by T* , if for all $x \in G$, we have $x \geq 0$ if and only if $\sigma(x) \geq 0$ for all $\sigma \in T$.

Lemma 3.6. *Let G be a partially ordered abelian group such that G^+ has finitely many atoms $\{x_1, x_2, \dots, x_k\}$ and $u = \sum_{j=1}^k x_j$ is an order unit. Suppose $G = \mathbb{Z}\langle x_1, x_2, \dots, x_k \rangle \oplus \ker \rho_G$*

as a group. Let $n > 1$ be an integer and let H be an interpolation group and order units v_i for $i = 1, 2, \dots, n$ such that the order of H is determined by $T \subseteq S(H, \sum_{i=1}^n v_i)$.

For any normalized positive group homomorphism $\alpha: (G, u) \rightarrow (H, \sum_{i=1}^n v_i)$, there exist normalized, positive group homomorphisms $\alpha_i: (G, u) \rightarrow (H, v_i)$ for $i = 1, 2, \dots, n$ such that $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_n$ and $\ker \rho_G \subseteq \ker \alpha_i$ for $i > 1$.

Furthermore, if H has strict interpolation and $\ker \alpha \cap \mathbb{Z}\langle x_1, x_2, \dots, x_k \rangle = 0$, then we can arrange it so that $\ker \alpha_i \cap \mathbb{Z}\langle x_1, x_2, \dots, x_k \rangle = 0$ for all i .

Proof. Since $\alpha(x_1) + \alpha(x_2) + \dots + \alpha(x_k) = \alpha(u) = v_1 + v_2 + \dots + v_n$, by the Riesz interpolation property, there exist $z_{i,j} \in H^+$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, k$ such that

$$\sum_{i=1}^n z_{i,j} = \alpha(x_j) \text{ and } \sum_{j=1}^k z_{i,j} = v_i$$

We define $\alpha_i: G \rightarrow H$ by setting $\alpha_i(x_j) = z_{i,j}$ for all i and j , setting $\alpha_1(g) = \alpha(g)$ for $g \in \ker \rho_G$, and setting $\alpha_i(g) = 0$ for $g \in \ker \rho$ and $i > 1$. Since the set of atoms is \mathbb{Z} -independent (Lemma 3.10 of [2]), α_i are group homomorphisms. By construction, $\ker \rho_G \subseteq \ker \alpha_i$ for $i > 1$.

Since $\sum_{i=1}^n \alpha_i = \alpha_1 = \alpha$ on $\ker \rho_G$ and $\sum_{i=1}^n \alpha_i(x_j) = \sum_{i=1}^n z_{i,j} = \alpha(x_j)$, we have $\sum_{i=1}^n \alpha_i = \alpha$.

Let $x \in G^+$. So there exist non-negative integers m_j for $j = 1, 2, \dots, k$ and $g \in \ker \rho$ so that $x = g + \sum_{j=1}^k m_j x_j$. Take $\tau \in T$. Since $\tau \circ \alpha_i \in S(G, u)$, we have $\tau(\alpha_i(g)) = 0$ for all i and so $\tau(\alpha_i(x)) = \sum_{j=1}^n m_j \tau(z_{i,j}) \geq 0$. Since the order of H is determined by T , we have that $\alpha_i(x) \geq 0$, and so α_i are positive group homomorphisms. Also $\alpha_i(u) = \alpha_i(\sum_{j=1}^k x_j) = \sum_{j=1}^k \alpha_i(x_j) = \sum_{j=1}^k z_{i,j} = v_i$. So $\alpha_i: (G, u) \rightarrow (H, v_i)$ are normalized positive group homomorphisms.

Suppose that G has strict interpolation and $\ker \alpha \cap \mathbb{Z}\langle x_1, x_2, \dots, x_k \rangle = 0$. Then $\alpha(x_j) > 0$ and since $v_i > 0$, by strict comparison, we can arrange $z_{i,j} > 0$. And so $\ker \alpha_i \cap \mathbb{Z}\langle x_1, x_2, \dots, x_k \rangle = 0$ for all i . \square

Lemma 3.7. *Let G_1 and G_2 be partially ordered abelian groups such that G_1^+ has finitely many atoms $\{x_1, x_2, \dots, x_k\}$ and G_2^+ has finitely many atoms $\{y_1, y_2, \dots, y_m\}$, where $u_1 = \sum_{j=1}^k x_j$ and $u_2 = \sum_{t=1}^m y_t$ are order units. Suppose that $G_1 = \mathbb{Z}\langle x_1, x_2, \dots, x_k \rangle \oplus \ker \rho_{G_1}$ and $G_2 = \mathbb{Z}\langle y_1, y_2, \dots, y_m \rangle \oplus \ker \rho_{G_2}$ as groups. Let $n > 1$ be an integer and let H be an interpolation group with order units v_i for $i = 1, 2, \dots, n$ such that the order of H is determined by $T \subseteq S(H, \sum_{i=1}^n v_i)$.*

Let $\alpha: (G_1, u_1) \rightarrow (G_2, u_2)$ be a normalized positive group homomorphism such that $\alpha(\mathbb{Z}\langle x_1, \dots, x_k \rangle) \subseteq \mathbb{Z}\langle y_1, y_2, \dots, y_m \rangle$. Let $\beta_s: (G_s, u_s) \rightarrow (H, \sum_{i=1}^n v_i)$ be normalized positive group homomorphisms for $s = 1, 2$ such that $\beta_1 = \alpha \circ \beta_2$. If there exist $\beta_{1,i}: (G_1, u_1) \rightarrow (H, v_i)$ such that $\sum_{i=1}^n \beta_{1,i} = \beta_1$ and $\ker \rho_{G_1} \subseteq \ker \beta_{1,i}$ for $i > 1$, then there exist $\beta_{2,i}: (G_2, u_2) \rightarrow (H, v_i)$ such that $\sum_{i=1}^n \beta_{2,i} = \beta_2$, $\ker \rho_{G_2} \subseteq \ker \beta_{2,i}$ for $i > 1$, and $\beta_{1,i} = \beta_{2,i} \circ \alpha$ for all i .

Furthermore, if H has strict interpolation, $\alpha(x_j) \neq 0$ for $j = 1, 2, \dots, k$, if $\ker \beta_2 \cap \mathbb{Z}\langle y_1, \dots, y_m \rangle = 0$, and if $\ker \beta_{1,i} \cap \mathbb{Z}\langle x_1, \dots, x_k \rangle = 0$ for all i , then we can arrange it so that $\ker \beta_{2,i} \cap \mathbb{Z}\langle y_1, \dots, y_m \rangle = 0$ for all i .

Proof. Since α is a positive homomorphism, $\alpha(u_1) = u_2$, and $\alpha(\mathbb{Z}\langle x_1, \dots, x_k \rangle) \subseteq \mathbb{Z}\langle y_1, \dots, y_m \rangle$, for each $j = 1, 2, \dots, k$ there exists a subset $S_j \subseteq \{1, 2, \dots, m\}$ such that $\alpha(x_j) = \sum_{t \in S(j)} y_t$. Furthermore, $S_i \cap S_j = \emptyset$ if $i \neq j$ and $\bigcup_{j=1}^k S_j = \{1, 2, \dots, m\}$. So we have

$$\sum_{t \in S_j} \beta_2(y_t) = \beta_2(\alpha(x_j)) = \beta_1(x_j) = \sum_{i=1}^n \beta_{1,i}(x_j).$$

By the Riesz interpolation property, there exist $z_{i,t} \in H^+$ for $t \in S_j$ and $i = 1, 2, \dots, n$ so that

$$\sum_{t \in S_j} z_{i,t} = \beta_{1,i}(x_j) \text{ and } \sum_{i=1}^n z_{i,t} = \beta_2(y_t).$$

We define $\beta_{2,i}: G_2 \rightarrow H$ by setting $\beta_{2,i}(y_t) = z_{i,t}$ for all i, t , setting $\beta_{2,1}(g) = \beta_2(g)$ for $g \in \ker \rho_{G_2}$ and setting $\beta_{2,i}(g) = 0$ for $g \in \ker \rho_{G_2}$ and $i > 1$. We see that $\beta_{2,i}$ are well-defined since the sets S_j partition $\{1, 2, \dots, m\}$ and $\beta_{2,i}$ are group homomorphisms since atoms are \mathbb{Z} -independent. By construction, $\ker \rho_{G_2} \subseteq \ker \beta_{2,i}$ for $i > 1$.

As before, $\sum_{i=1}^n \beta_{2,i} = \beta_{2,1} = \beta_2$ on $\ker \rho_{G_2}$ and $\sum_{i=1}^n \beta_{2,i}(y_t) = \sum_{i=1}^n z_{i,t} = \beta_2(y_t)$. So $\sum_{i=1}^n \beta_{2,i} = \beta_2$.

Notice that for all $\sigma \in S(G_2, u_2)$, we have $\sigma \circ \alpha \in S(G_1, u_1)$, so if $g \in \ker \rho_{G_1}$, then $\sigma \circ \alpha(g) = 0$ for all $\sigma \in S(G_2, u_2)$. So $\alpha(g) \in \ker \rho_{G_2}$. So on $\ker \rho_{G_1}$, $\beta_{1,i} = 0 = \beta_{2,i} \circ \alpha$ when $i > 1$ and $\beta_{1,1} = \beta_1 = \beta_2 \circ \alpha = \beta_{2,1} \circ \alpha$. Also $\beta_{2,i}(\alpha(x_j)) = \sum_{t \in S_j} \beta_{2,i}(y_t) = \sum_{t \in S_j} z_{i,t} = \beta_{1,i}(x_j)$ for all i, j . Thus $\beta_{1,i} = \beta_{2,i} \circ \alpha$ for all i . Further, since $\alpha(u_1) = u_2$, we have $\beta_{2,i}(u_2) = \beta_{1,i}(u_1) = v_i$.

Let $x \in G_2^+$. So there exist non-negative integers r_t and $g \in \ker \rho_{G_2}$ so that $x = g + \sum_{t=1}^m r_t y_t$. Take $\tau \in T$. Since $\tau \circ \beta_{2,i} \in S(G, u)$, we see $\tau(\beta_{2,i}(g)) = 0$ and so $\tau(\beta_{2,i}(x)) = \sum_{t=1}^m r_t \tau(z_{i,t}) \geq 0$. So $\beta_{2,i}$ are normalized positive group homomorphisms.

we can arrange it so that $\ker \beta_{2,i} \cap \mathbb{Z}\langle y_1, \dots, y_m \rangle = 0$ for all i .

Now assume H has strict interpolation, that $\alpha(x_j) \neq 0$ for $j = 1, 2, \dots, k$, that $\ker \beta_2 \cap \mathbb{Z}\langle y_1, \dots, y_m \rangle = 0$, and that $\ker \beta_i \cap \mathbb{Z}\langle x_1, \dots, x_k \rangle = 0$ for all i . It follows that all of the S_j are non-empty, that $\beta_2(y_t) > 0$ and $\beta_{1,i}(x_j) > 0$ for all i, j, t . So from strict interpolation, we can arrange for $z_{i,t} > 0$. So $\ker \beta_{2,i} \cap \mathbb{Z}\langle y_1, \dots, y_m \rangle = 0$ for all i . \square

4. HOMOMORPHISMS INTO C^* -ALGEBRAS OF TRACIAL RANK ONE

We recall a uniqueness theorem of Lin's:

Theorem 4.1. *Let C be a unital AH-algebra and A be a separable, simple unital C^* -algebra with tracial rank at most one. Let $\phi: C \rightarrow A$ be a unital, injective homomorphism. For every $\varepsilon > 0$, every finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$, a finite subset $\mathcal{P} \subseteq \underline{K}(C)$, a*

finite subset $\mathcal{U} \subseteq U_\infty(C)$, and a finite subset $\mathcal{G} \subseteq C$ such that for any unital homomorphism $\psi: C \rightarrow A$, if

1. $KL(\phi) = KL(\psi)$ on \mathcal{P} ,
2. $\text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta$ for $\bar{z} \in \mathcal{U}$, and
3. $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for $g \in \mathcal{G}$,

then there exists a unitary $u \in A$ such that

$$\|u\phi(f)u^* - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

This is simply Corollary 11.6 of [6] dropping the condition that C has Property (J). The same proof works in light of Theorem 5.8 and Lemma 5.7(2) of [8].

Theorem 4.2. *Let X be a compact metric space. Let A be a separable, simple, unital C^* -algebra with tracial rank at most one. Let $n \geq 1$ be an integer and let $\phi: C(X) \rightarrow M_n(A)$ be a unital, injective homomorphism. For any $\varepsilon > 0$ and any finite set $\mathcal{F} \subseteq C(X)$, there exist n unital homomorphisms $\phi_i: C(X) \rightarrow A$ for $i = 1, 2, \dots, n$ and a unitary $u \in M_n(A)$ such that for all $f \in \mathcal{F}$,*

$$\|u\phi(f)u^* - \text{diag}(\phi_1(f), \phi_2(f), \dots, \phi_n(f))\| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $\mathcal{F} \subseteq C(X)$ be a finite subset. By theorem 4.1, there exist $\delta > 0$, a finite subset $\mathcal{F} \subseteq C(X)$, a finite subset $\mathcal{P} \subseteq \underline{K}(C(X))$, and a finite subset $\mathcal{U} \subseteq U^\infty(C(X))$ such that for any unital homomorphism $\psi: C(X) \rightarrow M_n(A)$, if

1. $KL(\phi) = KL(\psi)$ on \mathcal{P} ,
2. $\text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta$ for $\bar{z} \in \mathcal{U}$, and
3. $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for $g \in \mathcal{G}$,

then there exists a unitary $u \in A$ such that

$$\|u\phi(f)u^* - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Since A has tracial rank at most one, the order of $K_0(A)$ is determined by the tracial states (Theorem 3.7.2 of [9]) and $K_0(A)$ has Riesz interpolation property (Theorem 6.11 of [10]).

First we consider the case where $K_0(A) \cong \mathbb{Z}$. Since ϕ is injective, $\ker K_0(\phi) \cap C(X, \mathbb{Z}) = 0$, and so $C(X, \mathbb{Z})$ is cyclic, meaning X is connected and $C(X, \mathbb{Z})$ is generated by $1_{C(X)}$, the only atom of $K_0(C(X))^+$. Since $K_0(C(X)) = C(X, \mathbb{Z}) \oplus \ker \rho_{C(X)}$ and since $K_0(1_{M_n(A)}) = n \cdot K_0(1_A)$, by Lemma 3.6, there exist normalized, positive group homomorphisms $\alpha_i: (K_0(C(X)), 1_{C(X)}) \rightarrow (K_0(A), 1_A)$ such that $K_0(\phi) = \sum_{i=1}^n \alpha_i$.

Let $\gamma_i: C(X)_{\text{sa}} \rightarrow \text{Aff}(T(A))$ be defined by $\gamma_i(f)(\tau) = \tau(\phi(f))$. We see that γ_i are positive, unital linear maps. Since ϕ is injective and A is simple, γ_i are strictly positive. Since X is connected, α_i and γ_i are vacuously compatible.

$\ker K_0(\phi) \cap C(X, \mathbb{Z}) = 0$, we see that $C(X, \mathbb{Z})$ is cyclic and so X is connected. By Lemma 3.6, there exist normalized, positive group homomorphisms $\alpha_i: K_0(C(X)) \rightarrow K_0(A)$ so that $K_0(\phi) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\ker \rho_A \subseteq \ker \alpha_i$ for $i > 1$.

We define positive, unital linear maps $\gamma_i: C(X)_{\text{sa}} \rightarrow \text{Aff}(T(A))$ by $\gamma_i(f)(\tau) = \tau \otimes \text{Tr}(\phi(f))$. We see that γ_i are strictly positive since ϕ is injective and A is simple. Also α_i and γ_i are trivially compatible.

We define group homomorphisms $\beta_i: K_1(C(X)) \rightarrow K_1(A)$ by $\beta_1 = K_1(\phi)$ and $\beta_i = 0$ for $i = 2, 3, \dots, n$. So $\sum_{i=1}^n \beta_i = \beta_1 = K_1(\phi)$.

For $1 < i \leq n$, we define $\kappa_i \in KL(C(X), A)$ so that $\Gamma(\kappa_i) = (\alpha_i, \beta_i)$. We set $\kappa_1 = KL(\phi) - \sum_{i=2}^n \kappa_i$. Notice that $\Gamma(\kappa_1) = K_*(\phi) - \sum_{i=2}^n (\alpha_i, \beta_i) = (\alpha_1, \beta_1)$. So we have $\kappa_i \in KL_e(C(X), A)^{++}$ compatible with γ_i and $\sum_{i=1}^n \kappa_i = KL(\phi)$.

The compatible pairs α_i and γ_i induce group homomorphisms

$$\eta_i^0: \text{Aff}(T(C(X)))/\rho_{C(X)}(K_0(C(X))) \rightarrow \text{Aff}(T(A))/\rho_A(K_0(A)).$$

We extend these homomorphisms to $\eta_i: U^\infty(C(X))/\overline{CU^\infty(C(X))} \rightarrow U(A)/\overline{CU(A)}$ by setting $\eta_1 = \phi^\dagger$ on $K_1(C(X))$ and $\eta_i = 0$ on $K_1(C(X))$ for $i > 1$. By naturality, we have $\pi_A \circ \phi^\dagger = K_1(\phi) \circ \pi_{C(X)}$, so by construction $\pi_A \circ \eta_1 = \beta_1 \circ \pi_{C(X)}$ and so η_1 and β_1 are compatible. Since $\beta_i = 0 = \eta_i$ on $K_1(C(X))$ for $i = 2, 3, \dots, n$, κ_i is compatible with η_i for $i = 2, 3, \dots, n$. By construction, η_i and γ_i are compatible. So for each i , we see that κ_i , γ_i , and η_i are compatible.

So by Theorem 6.10 of [8], there exist unital homomorphisms $\psi_i: C(X) \rightarrow A$ for $i = 1, 2, \dots, n$ such that

$$KL(\psi_i) = \kappa_i, \tau(\psi_i(f)) = \gamma_i(f)(\tau), \text{ and } \psi_i^\dagger = \eta_i.$$

Let $\psi = \text{diag}(\phi_1, \phi_2, \dots, \phi_n)$. We can see that

$$\begin{aligned} KL(\psi) &= \sum_{i=1}^n KL(\phi_i) = \sum_{i=1}^n \kappa_i = KL(\phi) \\ \tau(\psi(f)) &= \sum_{i=1}^n \tau(\phi_i(f)) = \sum_{i=1}^n \gamma_i(f)(\tau) = \tau(\phi(f)) \\ \psi^\dagger &= \sum_{i=1}^n \phi_i^\dagger = \sum_{i=1}^n \eta_i = \phi^\dagger \end{aligned}$$

So by Theorem 5.10 of [8], ϕ and ψ are approximately unitarily equivalent, and so ϕ is approximately diagonalizable.

Now suppose that $K_0(A)$ is not cyclic. Since $K_0(A)$ is a simple interpolation group, $K_0(A)$ has strict interpolation.

Since X is a compact metric space, there exist finite simplicial complexes X_m for $m \in \mathbb{N}$ and unital homomorphisms $s_m: C(X_m) \rightarrow C(X_{m+1})$ such that $C(X) \cong \lim_{m \rightarrow \infty} C(X_m)$. We assume, without loss of generality, that $X_{m+1} \subseteq X_m$ and that s_m are the homomorphisms induced by inclusion. Let $s_{m,\infty}: C(X_m) \rightarrow C(X)$ denote the homomorphisms induced by

the inductive limit. Let $k(m)$ denote the number of connected components of X_m and χ_m^j the characteristic functions of the connected components of X_m for $j = 1, 2, \dots, k(m)$. We may further assume that $s_{m,\infty}(\chi_m^j) \neq 0$ for all j .

Since \mathcal{G} is finite, there exist an integer M and a finite set $\mathcal{G}' \subseteq C(X_M)_{\text{sa}}$ such that for every $g \in \mathcal{G}$, there exists $g' \in \mathcal{G}'$ such that $\|g - s_{M,\infty}(g')\| < \delta/2$.

Furthermore, by taking a possibly larger value of M , there exists a finite set $\mathcal{U}' \subseteq U^\infty(C(X_M))/CU^\infty(C(X_M))$ such that for every $u \in \mathcal{U}$, there exists $u' \in \mathcal{U}'$ such that $\text{dist}(\bar{u}, s_{M,\infty}^\dagger(\bar{u}')) < \delta/2$.

Since X_M has finitely many connected components, $C(X_M, \mathbb{Z})$ is generated by the atoms of $K_0(C(X_M))^+$ and so $K_0(C(X_M)) = C(X_M, \mathbb{Z}) \oplus \ker \rho_{C(X_M)}$ is the decomposition necessary for the Lemmas 3.6 and 3.7. So by Lemma 3.6, there exist normalized group homomorphisms $\alpha_{i,M}: (K_0(C(X_M)), 1_{C(X_M)}) \rightarrow (K_0(A), 1_A)$ such that $K_0(\phi \circ s_{M,\infty}) = \sum_{i=1}^n \alpha_{i,M}$ and $\ker \alpha_{i,M} \cap C(X_M, \mathbb{Z}) = 0$ for all i with $\ker \alpha_{i,M} = \ker \rho_{C(X_M)}$ when $i > 1$.

Since A has stable rank one (Theorem 3.6.10 of [9]), there exist non-zero, mutually orthogonal projections $p_{i,j}^M \in M_n(A)$ for $i = 1, 2, \dots, n, j = 1, 2, \dots, k(M)$ such that $K_0(p_{i,j}^M) = \alpha_{i,M}(\chi_M^j)$ and $\sum_{i=1}^n p_{i,j}^M = \phi(s_{M,\infty}(\chi_M^j))$. We define $\gamma_{i,M}: C(X_M)_{\text{sa}} \rightarrow \text{Aff}(T(A))$ such that

$$\gamma_{i,M}(f)(\tau) = \sum_{j=1}^{k(M)} \tau(p_{i,j}^M \phi \circ s_{M,\infty}(f) p_{i,j}^M)$$

Since $p_{i,j}^M$ are non-zero, mutually orthogonal projections, $\gamma_{i,M}$ are positive, linear maps with $\ker \gamma_{i,M} = \ker s_{M,\infty}$. For all $\tau \in T(A)$ and j_0 , we have

$$\gamma_{i,M}(\chi_M^{j_0})(\tau) = \sum_{j=1}^{k(M)} \tau(p_{i,j}^M \phi(s_{M,\infty}(\chi_M^{j_0})) p_{i,j}^M) = \tau(p_{i,j_0}^M) = \tau(\rho_A(\alpha_i(\chi_M^{j_0})))$$

and so $\gamma_{i,M}$ and $\alpha_{i,M}$ are compatible.

We inductively apply Lemma 3.7 to construct normalized, positive group homomorphisms $\alpha_{i,m}: K_0(C(X_m)) \rightarrow K_0(A)$ for $i = 1, 2, \dots, n$ and $m \geq M$ so that $K_0(\phi \circ s_{m,\infty}) = \sum_{i=1}^n \alpha_{i,m}$ with $\alpha_{i,m} = \alpha_{i,m+1} \circ s_m$, and $\ker \alpha_{i,m} \cap C(X_m, \mathbb{Z}) = 0$ for all i with $\ker \alpha_{i,m} = \ker \rho_{C(X_m)}$ when $i > 1$.

As before, there exist non-zero, mutually orthogonal projections $p_{i,j}^m \in M_n(A)$ for $i = 1, 2, \dots, n, j = 1, 2, \dots, k(m)$ such that $K_0(p_{i,j}^m) = \alpha_{i,m}(\chi_m^j)$ and $\sum_{i=1}^n p_{i,j}^m = \phi \circ s_{m,\infty}(\chi_m^j)$. We see that $\gamma_{i,m}$ are positive unital linear maps with $\ker \gamma_{i,m} = \ker s_{m,\infty}$. The computation that $\gamma_{i,m}$ and $\alpha_{i,m}$ are compatible is identical to the case where $m = M$.

Let $\alpha_i = \lim_{m \rightarrow \infty} \alpha_{i,m}$ and $\gamma_i = \lim_{m \rightarrow \infty} \gamma_{i,m}$. Since $\sum_{i=1}^n \alpha_{i,m} = K_0(\phi \circ s_{m,\infty})$, $\sum_{i=1}^n \alpha_i = K_0(\phi)$. Since $\ker \alpha_{i,m} \cap C(X_m, \mathbb{Z}) = 0$, it follows that $\ker \alpha_i \cap C(X, \mathbb{Z}) = 0$ for all i . Also γ_i are injective, since $\ker \gamma_{i,m} = \ker s_{m,\infty}$. And since $\gamma_{i,m}$ and $\alpha_{i,m}$ are compatible, γ_i and α_i are compatible.

We define group homomorphisms $\beta_i: K_1(C(X)) \rightarrow K_1(A)$ by $\beta_1 = K_1(\phi)$ and $\beta_i = 0$ for $i = 2, 3, \dots, n$. So $\sum_{i=1}^n \beta_i = \beta_1 = K_1(\phi)$.

For $1 < i \leq n$, we define $\kappa_i \in KL(C(X), A)$ so that $\Gamma(\kappa_i) = (\alpha_i, \beta_i)$. We set $\kappa_1 = KL(\phi) - \sum_{i=2}^n \kappa_i$. Notice that $\Gamma(\kappa_1) = K_*(\phi) - \sum_{i=2}^n (\alpha_i, \beta_i) = (\alpha_1, \beta_1)$. So we have $\kappa_i \in KL_e(C(X), A)^{++}$ compatible with γ_i and $\sum_{i=1}^n \kappa_i = KL(\phi)$.

The compatible pairs α_i and γ_i induce group homomorphisms

$$\eta_i^0: \text{Aff}(T(C(X))/\rho_{C(X)}(K_0(C(X)))) \rightarrow \text{Aff}(T(A))/\rho_A(K_0(A)).$$

We extend these homomorphisms to $\eta_i: U^\infty(C(X))/\overline{CU^\infty(C(X))} \rightarrow U(A)/\overline{CU(A)}$ by setting $\eta_1 = \phi^\dagger$ on $K_1(C(X))$ and $\eta_i = 0$ on $K_1(C(X))$ for $i > 1$. By naturality, we have $\pi_A \circ \phi^\dagger = K_1(\phi) \circ \pi_{C(X)}$, so by construction $\pi_A \circ \eta_1 = \beta_1 \circ \pi_{C(X)}$ and so η_1 and β_1 are compatible. Since $\beta_i = 0 = \eta_i$ on $K_1(C(X))$ for $i = 2, 3, \dots, n$, κ_i is compatible with η_i for $i = 2, 3, \dots, n$. By construction, η_i and γ_i are compatible. So for each i , we see that κ_i , γ_i , and η_i are compatible. Since on $K_1(C(X))$, we have $\sum_{i=1}^n \eta_i = \phi^\dagger$, we have $\sum_{i=1}^n \eta_i \circ s_{M,\infty}^\dagger = (\phi \circ s_{M,\infty})^\dagger$ on $K_1(C(X_M))$. Also since η_i are induced from γ_i on $\text{Aff}(T(C(X))/\rho_{C(X)}(K_0(C(X))))$, we have $\sum_{i=1}^n \eta_i \circ s_{M,\infty}^\dagger = (\phi \circ s_{M,\infty})^\dagger$. So we have $\sum_{i=1}^n \eta_i \circ s_{M,\infty}^\dagger = (\phi \circ s_{M,\infty})^\dagger$ on $U^\infty(C(X_M))/\overline{CU(C(X_M))}$.

By Theorem 4.5 of [8], there exist unital injective homomorphisms $\phi_i: C(X) \rightarrow A$ such that

$$KL(\phi_i) = \kappa_i, \tau(\phi_i(f)) = \gamma_i(f)(\tau), \text{ and } \phi_i^\dagger = \eta_i$$

for all $f \in C(X)_{\text{sa}}$ and $\tau \in T(A)$. Let $\psi = \text{diag}(\phi_1, \phi_2, \dots, \phi_N)$. So $KL(\psi) = \sum_{i=1}^n KL(\phi_i) = \sum_{i=1}^n \kappa_i = KL(\phi)$. In particular, this holds for \mathcal{P} .

Let $f \in \mathcal{G}$ and $\tau \in T(M_n(A))$, there exists $f' \in \mathcal{G}'$ so that $\|f - s_{M,\infty}(f')\| < \delta/2$. Note that

$$\begin{aligned} \tau(\psi(s_{M,\infty}(f'))) &= \sum_{i=1}^n \gamma_i(s_{M,\infty}(f'))(\tau) \\ &= \sum_{i=1}^n \sum_{j=1}^{k(M)} \tau(p_{i,j}^M \phi(s_{M,\infty}(f')) p_{i,j}^M) \\ &= \sum_{j=1}^{k(M)} \tau(\phi(s_{M,\infty}(\chi_M^j)) \phi(s_{M,\infty}(f')) \phi(s_{M,\infty}(\chi_M^j))) \\ &= \tau(\phi(s_{M,\infty}(f'))). \end{aligned}$$

Consequently,

$$\begin{aligned} |\tau(\phi(f)) - \tau(\psi(f))| &\leq |\tau(\phi(f)) - \tau(\phi(s_{M,\infty}(f')))| \\ &\quad + |\tau(\phi(s_{M,\infty}(f')) - \tau(\psi(s_{M,\infty}(f')))| \\ &\quad + |\tau(\psi(s_{M,\infty}(f')) - \tau(\psi(f))| \\ &< \|\tau \circ \phi\| (\delta/2) + \|\tau \circ \psi\| (\delta/2) = \delta. \end{aligned}$$

Let $u \in \mathcal{U}$. There exists $u' \in \mathcal{U}'$ such that $\text{dist}(\bar{u}, s_{M,\infty}^\dagger(\bar{u}') < \delta/2$. So we have

$$\begin{aligned} \text{dist}(\phi^\dagger(\bar{u}), \psi^\dagger(\bar{u})) &\leq \text{dist}(\phi^\dagger(\bar{u}), (\phi \circ s_{M,\infty})(\bar{u})) \\ &\quad + \text{dist}((\phi \circ s_{M,\infty})^\dagger(\bar{u}), (\psi \circ s_{M,\infty})^\dagger(\bar{u})) \\ &\quad + \text{dist}((\psi \circ s_{M,\infty})^\dagger(\bar{u}), \psi^\dagger(\bar{u})) \\ &\leq \delta/2 + 0 + \delta/2 = \delta. \end{aligned}$$

Therefore, there exists a unitary $u \in M_n(A)$ such that for all $f \in \mathcal{F}$,

$$\|u\phi(f)u^* - \text{diag}(\phi_1(f), \phi_2(f), \dots, \phi_n(f))\| < \varepsilon. \quad \square$$

Corollary 4.3. *Let X be a compact metric space and let A be a simple, separable, unital C^* -algebra with tracial rank at most one. Let $n \geq 1$ be an integer and $\phi: C(X) \rightarrow M_n(A)$ be a unital homomorphism. For any $\varepsilon > 0$ and any finite set $\mathcal{F} \subseteq C(X)$, there exist n unital homomorphisms $\phi_i: C(X) \rightarrow A$ for $i = 1, 2, \dots, n$ and a unitary $u \in M_n(A)$ such that for all $f \in \mathcal{F}$,*

$$\|u\phi(f)u^* - \text{diag}(\phi_1(f), \phi_2(f), \dots, \phi_n(f))\| < \varepsilon.$$

Proof. There exists a metric space Y such that $C(Y) \cong C(X)/\ker \phi$. Let $\psi: C(Y) \rightarrow M_n(A)$ denote the induced injective homomorphism and $\pi: C(X) \rightarrow C(Y)$ denote the canonical quotient. By the theorem, there exist unital homomorphisms $\psi_n: C(Y) \rightarrow M_n(A)$ and unitary $u \in M_n(A)$ such that

$$\|u\psi(g)u^* - \text{diag}(\psi_1(g), \psi_2(g), \dots, \psi_n(g))\| < \varepsilon$$

for all $g \in \pi(\mathcal{F})$. So for all $f \in \mathcal{F}$,

$$\|u\psi(\pi(f))u^* - \text{diag}(\psi_1(\pi(f)), \psi_2(\pi(f)), \dots, \psi_n(\pi(f)))\| < \varepsilon$$

So by setting $\phi_i = \psi_i \circ \pi$, we obtain the result. \square

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